COMBINATORICA 4 (4) (1984) 265—273

CYCLES THROUGH TEN VERTICES IN 3-CONNECTED CUBIC GRAPHS

M. N. ELLINGHAM, D. A. HOLTON and C. H. C. LITTLE

Received 10 April 1983

It is known that there exists a cycle through any nine vertices of a 3-connected cubic graph G. Here we show that if an edge is removed from such a graph, then there is still a cycle through any five vertices. Furthermore, we characterise the circumstances in which there fails to be a cycle through six. As corollaries we are able to prove that a 3-connected cubic graph has a cycle through any specified five vertices and one edge, and to classify the conditions under which it has a cycle through four chosen vertices and two edges.

We are able to use the five and six vertex results to show that a 3-connected cubic graph has a cycle which passes through any ten given vertices if and only if the graph is not contractible to the Petersen graph in such a way that the ten vertices each map to a distinct vertex of the Petersen graph.

1. Introduction

In [4] it is shown that there exists a cycle through any nine vertices of a 3-connected cubic graph G. Here we show that if an edge is removed from such a graph, then there is still a cycle through any five vertices. Furthermore, we characterise the circumstances in which there fails to be a cycle through six. As corollaries we are able to prove that a 3-connected cubic graph has a cycle through any specified five vertices and one edge, and to classify the conditions under which it has a cycle through four chosen vertices and two edges.

These results were motivated by the conjecture in [4], that any 3-connected cubic graph either has a cycle through any 10 vertices, or is contractible to the Petersen graph. Using the facts proved here, in particular the 6 Vertices Excluding 1 Edge Theorem, we have been able to show that this conjecture is true.

Kelmans and Lomonosov [6] say that are also able to prove this result. Their methods are unknown to us at the moment, but may be related to those in [5].

Our proof of the conjecture basically follows and extends the proof of the "Nine Point Theorem" in [4], although it does not depend upon this theorem in any way.

AMS subject classification (1980): 05 C 40, 05 C 38

2. Definitions

Our graphs follow the definition given in [1]; they may have loops and multiple edges. For a given graph G we shall use φ_G to represent the incidence function of G, assigning a 1- or 2-vertex set to each edge of G.

Let G and H be graphs. A surjection $\gamma: VG \cup EG \to VH \cup EH$ (loosely written $\gamma: G \to H$) is a *contraction of* G *onto* H provided

(i) $\gamma(VG) = VH$;

(ii) For all $v \in VH$, $\gamma^{-1}(v)$ is a connected subgraph of G;

(iii) If $X = \gamma^{-1}(EH) \subseteqq EG$ then γ_X: $X \to EH$ is a bijection and $\gamma(\varphi_G(e)) = \varphi_H(\gamma(e))$ for all $e \in X$.

The graph H is also called a *contraction of* G. Here our definition is equivalent to the more usual one, namely a graph obtained by repeatedly deleting an edge and identifying the endvertices.

Let G be a graph and $S \subseteqq VG$. Then an edge $e \in EG$ is said to be *unavoidable given* S if every cycle C with $S \subseteqq VC$ has $e \in EC$.

Let G be a graph and $S \subseteqq VG$. Then the *coboundary of* S, denoted δS, is the set of all edges joining a vertex in S to a vertex in $VG - S$. If $|\delta S| = n$ then the subgraph H of G induced by S is called an *n-cut-subgraph of* G. If both H and the subgraph of G induced by $VG - S$ contain cycles then we say H is a *cyclic-n-cut-subgraph of* G. If G has no cyclic n-cut-subgraphs for any $n < k$ then we say G is *cyclically-k-edge-connected.*

Let G be a graph. We can define a distance function d_G not just on the vertices of G, but on the vertices and edges of G. If $u, v, w, x \in VG$ and $uv, wx \in EG$, then

$$d_G(u, w) = \text{the length of the shortest path from } u \text{ to } w \text{ in } G;$$

$$d_G(u, wx) = 1/2 + \min\,(d_G(u, x));$$

$$d_G(uv, wx) = \begin{cases} 1 + \min\,(d_G(u, w), d_G(u, x), d_G(v, w), d_G(v, x)), & uv \neq wx \\ 0, \; uv = wx. \end{cases}$$

The function d_G is a metric on $VG \cup EG$.

Let G be a cubic graph, and $X = \delta S = \{u_1v_1, u_2v_2, u_3v_3\}$ a coboundary consisting of three independent edges, with $u_1, u_2, u_3 \in S$ and $v_1, v_2, v_3 \in VG - S$. Let L and M be the subgraphs of G induced by S and $VG - S$ respectively. Then the cubic graphs H and J with $VH = VL \cup \{u\}$, $EH = EL \cup \{u_1u, u_2u, u_3u\}$ and $VJ = VM \cup \{v\}$, $EJ = EM \cup \{vv_1, vv_2, vv_3\}$ are called the *3-cut-reductions of* G *corresponding to* X. This concept is illustrated in Figure 2.1. A 3-cut-reduction of G is also a contraction of G: H is obtained by contracting the subgraph M to the single vertex u, and J by contracting L to v.

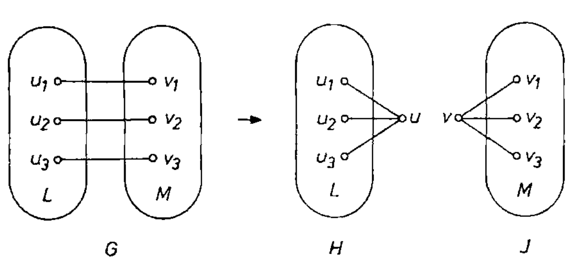

Fig. 2.1

Note that a 3-cut-reduction of a 3-connected cubic graph is always 3-connected.

Let G be a cubic graph and $e=xy\in EG$. If the neighbours of x are y, x_1, x_2, and the neighbours of y are x, y_1, y_2 then the *e-(edge)-reduction of* G is the cubic graph $H=G-\{x, y\}\cup\{x_1x_2, y_1y_2\}$. For each edge of $G-e$ there is a corresponding edge of H. For xx_1 or xx_2 it is x_1x_2, for yy_1 or yy_2 it is y_1y_2, and for any other edge it is the edge itself.

We note that an e-reduction of a cyclically-4-edge-connected cubic graph is 3-connected.

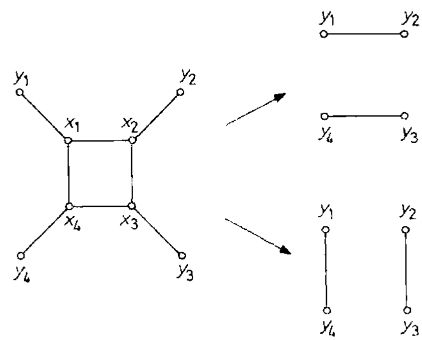

Fig. 2.2

Let G be a cubic graph containing a 4-cycle $C=x_1x_2x_3x_4x_1$. Suppose each x_i has a neighbour $y_i\notin VC$ (these need not all be distinct). Then the *4-cycle-reductions of G corresponding to C* are the graphs $G-VC\cup\{y_1y_2, y_3y_4\}$ and $G-VC\cup\cup\{y_1y_4, y_2y_3\}$ (see Figure 2.2).

We shall use freely the fact that a cubic graph with >2 vertices is k-connected if and only if it is k-edge-connected, for $k=1, 2$ or 3.

3. Cycles through specified vertices avoiding a given edge

Here we give the theorems concerning cycles through five and six specified vertices which avoid a specified edge.

Theorem 3.1 (The 5 Vertices Excluding 1 Edge Theorem). *Let G be a 3-connected cubic graph, and let $e\in EG$. Let $A\subseteq VG$ with $|A|\leqq 5$. Then there is a cycle C in G with $A\subseteq VC$, $e\notin EC$.* ▮

We omit the proof as it is very similar to that of Theorem 3.3.

Corollary 3.2. *Let H be a cyclic-4-cut-subgraph of a cyclically-4-edge-connected cubic graph. Suppose H has degree 2 vertices u_0, u_1, u_2 and u_3. Let H also be a subgraph of a cubic graph G, with edges $u_0v_0, u_1v_1, u_2v_2, u_3v_3\in EG$ where $v_0, v_1, v_2, v_3\in VG-VH$. For each $i\in\{1, 2, 3\}$ let G_i be the graph with $VG_i=VG-VH\cup\{w, x\}$ and $EG_i=E(G-VH)\cup\{wx, v_0x, v_ix, v_jw, v_kw\}$, where $(i, j, k)=\{1, 2, 3\}$.*

Let $y \in VH$. Then for at least two values of i the following is true: for every cycle G in G_i with $w \in VC$ there is a cycle D in G with $y \in VD$ and $D-VH= =C-\{w, x\}$. Such values of i are called *cyclically apt with respect to the replacement of H by wx*. We say that C in G_i extends to D in G.

The graphs P (the Petersen graph, displayed in Figure 3.1) and Q (depicted in Figure 3.2) play a central role in the rest of this paper. Whenever we refer to them, we shall assume that the vertices are labelled as in these figures. In the next theorem, we show that a graph G has unavoidable edges given A, where $|A|=6$, if and only if it is contractible to P or Q in a special way.

Theorem 3.3 (The 6 Vertices Excluding 1 Edge Theorem). *Let G be a 3-connected cubic graph, and let $A \subseteq VG$, $|A|=6$. Let X be the set of unavoidable edges given A. Then:*

(i)

$|X|=0, 1$ *or* 3.

(ii)

$|X|=1$ *if and only if there is a contraction* $\alpha\colon G \to Q$ *with* $\alpha(A)=B_Q= =\{a_1, a_2, b_5, b_6, b_7, b_8\}$; *then* $\alpha(X)=X_Q=\{a_1a_2\}$.

(iii)

$|X|=3$ *if and only if there is a contraction* $\beta\colon G \to P$ *with* $\beta(A)=B_p= =\{a_1, a_2, b_5, b_6, b_7, b_8\}$; *then* $\beta(X)=X_p=\{a_1a_2, b_5b_7, b_6b_8\}$.

(iv)

X is a set of independent edges.

If a contraction α or β, as described in (ii) or (iii) respectively, exists we shall call it a *canonical* contraction.

Proof. The proof is by induction on $|VG|$. The theorem is vacuously true when $|VG|=2$. Suppose $|VG|\geqq 4$ and the theorem holds for any 3-connected cubic graph G' with $|VG'|<|VG|$.

The proof proceeds as follows. We first show that if X is nonempty a canonical contraction exitst. Then we show that if α exists, $\alpha(X)=X_Q$ and hence $|X|=1$. Next we show that if β exists, $\beta(X)=X_P$ and thus $|X|=3$. It then follows immediately that X is a set of independent edges.

First we must show that if X is nonempty a canonical contraction exists. So let $e \in EG$. Then we search for a cycle through A avoiding e.

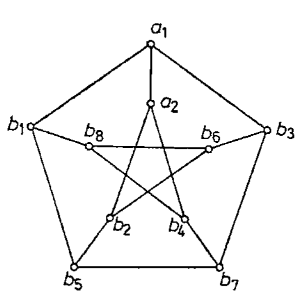

Fig. 3.1

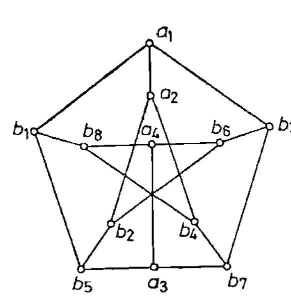

Fig. 3.2

(1) Suppose G is not cyclically-4-edge-connected. Then G contains a coboundary $\{u_1v_1, u_2v_2, u_3v_3\}$. Let H and J be the two corresponding 3-cut-reductions, with new vertices $u \in VH - VG$ adjacent to u_1, u_2 and u_3 in H, and $v \in VJ - VG$ adjacent to v_1, v_2 and v_3 in J. We may suppose that $e \in EH$ (if $e = u_iv_i$ for some i, we consider $e' = u_iu \in EH$ instead of e). Let $A_H = (A \cap VH) \cup \{u\}$ and $A_J = (A \cap VJ) \cup \{v\}$. If the required cycles exist through A_H and A_J, then these can be extended to cycles through A avoiding e. Otherwise G is contractible to P or Q in the required manner since G is contractible to H.

(2) Suppose G is cyclically-4-edge-connected.

(2.1) $|VG| \leqq 10$. These graphs can be found in [2]. They satisfy the theorem.

(2.2) $|VG| \geqq 12$.

(2.2.1) First suppose that every edge of G is incident to a vertex of A. Since $|A| = 6$ there are at most 18 edges incident to some vertex of A. Thus $|EG| \leqq 18$ and therefore $|VG| \leqq 12$. Hence $|VG| = 12$ and every edge is incident to exactly one vertex of A. G is therefore bipartite, with parts A and $VG - A$. The only cyclically-4-edge-connected cubic bipartite graphs with 12 vertices are those listed as 12.66, 12.73 and 12.81 in [2]. It can be verified that they all have a hamiltonian cycle avoiding any given edge $e \in EG$.

(2.2.2) Assume there exists $f \in EG$ incident to no vertex of A. Then the f-reduction H of G is 3-connected, and $A \subseteqq VH$. If $f = e$ then by induction H has a cycle C with $A \subseteqq VC$; C extends to the desired cycle in G through A avoiding e. Otherwise, let e_H be the edge in H corresponding to e in G. Let X_H be the set of unavoidable edges given A in H. If $e_H \notin X_H$ then there is a cycle C in H through A but not e_H, which induces the desired cycle in G. Assume therefore that $e_H \in X_H$. Since X_H is nonempty, by induction there is a contraction γ: $H \to J$ ($J = P$ or Q) such that $\gamma(A) = B_J$ and $\gamma(X_H) = X_J$. Let $e_J = \gamma(e_H)$. For each $v \in VJ$ let $S(v) = \gamma^{-1}(v)$, and let $T(v)$ be the corresponding subgraph of G. Suppose that $S(v)$ is nontrivial (that is, has more than one vertex). Let $\delta V(S(v)) = \{v_1x_1, v_2x_2, v_3x_3\}$, where $v_1, v_2, v_3 \in V(S(v))$. $T(v)$ must contain one end of $f = yz$, for otherwise $\delta V(S(v))$ is a cyclic-3-edge-cutset of G. Note that $z \notin \{x_1, x_2, x_3\}$, as otherwise G is not cyclically-4-edge-connected. Since yz has only two ends there are at most two vertices $v \in VJ$ for which $S(v)$ is nontrivial.

If $|S(v)| = 1$ for all $v \in VJ$ then $H = J$, so that G may be obtained by joining two vertices y and z inserted on nonidentical, nonadjacent (since G is cyclically-4-edge-connected) edges of J.

If there is exactly one v for which $|S(v)| > 1$, we can replace $T(v)$ by an edge vy joined to $G - V(T(v))$ by edges yz, yx_i, vx_j and vx_k, where $\{i, j, k\} = \{1, 2, 3\}$ to form a graph G_i. Then J is just the yz-reduction of G_i. Also, if e_{G_i} in G_i corresponds to e in G, then the edge obtained from e_{G_i} after the yz-reduction is just e_J. Further, any vertex $w \in A \cap T(v)$ maps to v under the contraction γ. Thus for at least two values of $i \in \{1, 2, 3\}$, every cycle in G_i through A_J avoiding e_{G_i} extends to a cycle of G using A but not e.

Otherwise there are two vertices $u, v \in VJ$ with $|S(u)|$, $|S(v)| > 1$. Then we can replace both $T(u)$ and $T(v)$ in G by edges uy and vz respectively, joined to each other by yz and to $G-V(T(u)\cup T(v))$ by edges yw_i, uw_j, uw_k and zx_p, vx_q, vx_r, where $\{i, j, k\}=\{p, q, r\}=\{1, 2, 3\}$ to form a graph G_{ip}. Once again we find J to be just the yz-reduction of G_{ip}; also if $e_{G_{ip}}$ in G_{ip} corresponds to e in G, then after the yz-reduction of G_{ip} we just obtain e_J from $e_{G_{ip}}$. Any vertex in $A\cap T(u)$ or $A\cap T(v)$ maps to u or v respectively under the contraction of H to J. So there are at least two cyclically apt values of i and at least two cyclically apt values of p such that every cycle in G_{ip} through A_J avoiding $e_{G_{ip}}$ extends to a cycle in G through A avoiding e.

An exhaustive study of the cases $J=P$ and $J=Q$ then shows that if X is nonempty a canonical contraction α or β exists.

Now we suppose that a contraction α: $G\to Q$ exists, with $\alpha(A)=B_Q$. We wish to show that $\alpha(X)=X_Q$. Suppose that $e\in EQ$. Let $N(v)=\alpha^{-1}(v)$ for $v\in VQ$.

Suppose that $\alpha(e)\in VQ$. Then $e\in E(N(u))$ for some $u\in VQ$. $N(u)$ must contain more than one vertex, and thus $\{u_1v_1, u_2v_2, u_3v_3\}=\delta V(N(u))$, where $u_1, u_2, u_3\in V(N(u))$, is a cyclic-3-edge-cutset of G. Now $N(u)$ contains at most one vertex of A, so that we are in the first part of the proof. But then a cycle through A avoiding e always existed, so that $e\notin X$.

Assume now that $\alpha(e)\in EQ-X_Q$. Then it can be shown that there is a cycle in Q through B_Q avoiding $\alpha(e)$, and hence a cycle through A avoiding e in G. There fore $e\notin X$.

Finally, suppose $\alpha(e)\in X_Q$. If there is a cycle C in G with $A\subseteq VC$ and $e\notin EC$, then $D=\alpha(C)$ is a cycle in Q with $B_Q=\alpha(A)\subseteq VD$ and $\alpha(e)\in ED$, which is impossible since $\alpha(e)\in X_Q$. Thus $e\in X$.

So we have shown that $e\in X$ if and only if $\alpha(e)\in X_Q$, or in other words $\alpha(X)=X_Q$. Since α maps exactly one edge to any given edge of Q, we now know that $|X|=|X_Q|=1$.

The argument showing that if a contraction β: $G\to P$ with $\beta(A)=B_P$ exists, then $\beta(X)=X_P$ and $|X|=3$, is completely analogous to the above. This completes the proof of the theorem. ∎

We have the following two corollaries.

Corollary 3.4: *Let G be a 3-connected cubic graph, and let $e\in EG$. Let $A\subseteq VG$ with $|A|\leqq 5$. Then there is a cycle C in G with $e\in EC$ and $A\subseteq VC$.* ∎

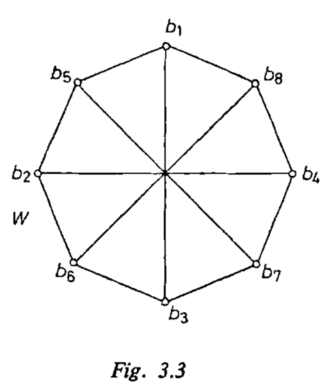

Fig. 3.3

Corollary 3.5. *Let G be a 3-connected cubic graph. Let $A \subseteqq VG$ with $|A| \leqq 4$, and let $e, f \in EG$. Then exactly one of the following is true:*

(i)

G contains a cycle C such that $A \subseteqq VC$ and $e, f \in EC$.

(ii)

There is a contraction $\mu: G \to P$ such that $\mu(A) = \{b_1, b_2, b_3, b_4\}$ and $\mu(\{e, f\}) = \{b_5 b_7, b_6 b_8\}$.

(iii)

There is a contraction $\nu: G \to W$ such that $\nu(A) = \{b_5, b_6, b_7, b_8\}$ and *$\nu(\{e, f\}) = \{b_1 b_3, b_2 b_4\}$, where W is the graph of Figure 3.3.*

4. The 10 vertex theorem

We now prove the main result of this paper. The method of attack is analogous to that of Theorems 3.1 and 3.3.

Theorem 4.1. (The 10 Vertex Theorem). *Let G be a 3-connected cubic graph and let $A \subseteqq VG$, $|A| \leqq 10$. Then there is a cycle C in G with $A \subseteqq VC$ if and only if there is no contraction $\gamma: G \to P$ such that $\gamma(A) = VP$.*

Proof. The proof is by induction on $|VG|$. If $|VG| \leqq 10$ then the theorem is true since all such graphs are hamiltonian except the Petersen graph which is hypohamiltonian. So now we may assume that $|VG| > 10$, and that the theorem is true for all 3-connected cubic graphs with less vertices than G.

($\Leftarrow$) Suppose there exists a contraction $\gamma: G \to P$ such that $\gamma(A) = VP$. If C is a cycle in G with $A \subseteqq VC$, then $D = \gamma(C)$ is a cycle in P with $VP \subseteqq VD$, or in other words a hamiltonian cycle in P. But P is nonhamiltonian, therefore no such cycle C can exist.

($\Rightarrow$) Now we must show that either a cycle C through A exists, or else G is contractible to P in such a way that A maps to VP. There are two cases.

(1) Suppose G is not cyclically-4-edge-connected. Assume first that $|A| = 10$. Then G contains a coboundary $\{u_1 v_1, u_2 v_2, u_3 v_3\}$. Let H and J be the two corresponding 3-cut-reductions, with new vertices $u \in VH - VG$ adjacent to u_1, u_2 and u_3 in H, and $v \in VJ - VG$ adjacent to v_1, v_2 and v_3 in J. As in Theorem 3.3 it is now striaghtforward to produce the required cycle from cycles in H and J or to show that G is contractible to P. When $|A| \leqq 9$ we can extend A to a set of 10 points; if these 10 points do not lie on a cycle then G is contractible to P and we can use the fact that P is hypohamiltonian to find a cycle through A.

(2) Suppose G is cyclically-4-edge-connected. In this case G cannot be contractible to P, so we are always seeking to prove that G has a cycle through A; we can assume (by adding extra vertices if necessary) that $|A| = 10$.

(2.1) Assume there exists $f \in EG$ incident to no vertex of A. Then the f-reduction H of G is 3-connected, and $A \subseteqq VH$. If H has a cycle through A then G also

clearly has a cycle through A, so suppose that H has no cycle through A. Then by induction there is a contraction $\gamma\colon H \to P$ such that $\gamma(A) = VP$.

For each $v \in VP$ let $S(v) = \gamma^{-1}(v)$, and let $T(v)$ be the corresponding subgraph of G. Suppose that $S(v)$ is nontrivial (that is, has more than one vertex). Let $\delta V(S(v)) = \{v_1x_1, v_2x_2, v_3x_3\}$, where $v_1, v_2, v_3 \in V(S(v))$. Then $T(v)$ must contain one end of $f = yz$, for otherwise $S(v)$ is a cyclic-3-cut-subgraph of G. Note that $z \notin \{x_1, x_2, x_3\}$, as otherwise G is not cyclically-4-edge-connected.

Since yz has only two ends there are at most two vertices $v \in VP$ for which $S(v)$ is nontrivial.

Let $s = |\{v \in VP\colon |V(S(v))| > 1\}|$. Suppose $s = 0$. Then $H = P$ and G can be obtained from P by subdividing two nonidentical, nonadjacent edges g and h to form new vertices y and z, and adding an edge yz. There are in fact only two possibilities for G, because edges g and h in P must be at distance 2 or distance 3. If they are at distance 2 we obtain the graph R of Figure 4.1 as G; if they are at distance 3 we obtain Q. But both Q and R are hamiltonian.

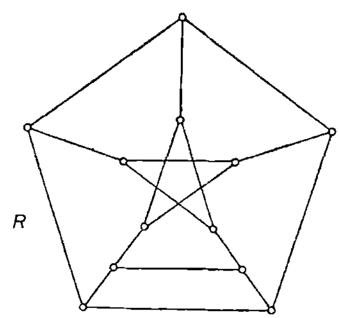

Fig. 4.1

The cases $s = 1$ and $s = 2$ are dismissed in a similar manner.

(2.2) Suppose now that every edge of G is incident to some vertex of A. Since each vertex in A is an endvertex of at most 3 edges we have $|EG| \leqq 3|A| \leqq 30$, which implies $|VG| \leqq 20$.

(2.2.1) $|VG| = 20$. In this case $|EG| = 3|A|$ so every edge of G must touch exactly one vertex of A. Thus G is bipartite, with parts A and $VG - A$. But then G is hamiltonian by [3], Theorem 7, and hence has a cycle through A.

(2.2.2) $12 \leqq |VG| \leqq 14$. Here G is hamiltonian.

(2.2.3) $16 \leqq |VG| \leqq 18$. The proof now breaks down into a tedious case analysis. If the girth of G is 4 then G has a 3-connected 4-cycle-reduction H. Since H is either hamiltonian with a cycle through at least one of any pair of edges or is contractible to P, G is either hamiltonian or it is one of five graphs (see [2]). Analysing the ways in which G can be obtained from these five graph shows that in all cases G is hamiltonian.

For G having girth greater than 4 we follow the approach of the corresponding case in the proof of the Nine Point Theorem in [4]. This time G is not necessarily always demonstrably hamiltonian, but always contains a cycle through any specified ten vertices. ∎

References

[1] J. A. BONDY and U. S. R. MURTY, *Graph Theory with Applications*, Macmillan, London, 1976.

[2] F. C. BUSSEMAKER, S. COBELJIĆ, D. M. CVETKOVIĆ and J. J. SEIDEL, Computer investigation of cubic graphs, *Technological University of Eindhoven, Mathematics Research Report WSK*·01, 1976.

[3] M. N. ELLINGHAM, Constructing certain cubic graphs, *Combinatorial Mathematics IX, Lecture Notes in Maths., No.* **952**, Springer, Berlin, 1982, 252—274.

[4] D. A. HOLTON, B. D. MCKAY, M. D. PLUMMER and C. THOMASSEN, A nine point theorem for 3-connected graphs, *Combinatorica*, **2** (1982), 53—62.

[5] A. K. KELMANS and M. V. LOMONOSOV, When m vertices in a k-connected graph cannot be walked round along a simple cycle, *Discrete Math.* **38** (1982), 317—322.

[6] A. K. KELMANS and M. V. LOMONOSOV, A cubic 3-connected graph having no cycles through given 10 vertices has the "Petersen" form, *Amer. Math. Soc. Abstracts, No. 82T-05-260*, **3** (1982), 283.

M. N. Ellingham,

Department of Combinatorics and Optimization,
University of Waterloo,
Waterloo,
Ontario N2L 3G1,
Canada.

D. A. Holton,

Department of Mathematics,
University of Melbourne,
Parkville,
Victoria 3052,
Australia.

C. H. C. Little,

Department of Mathematics and Statistics,
Massey University,
Palmerston North,
New Zealand.